Putnam training problems

2017 - Set 9

Problem 1 *Find all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *such that for all* $x, y \in \mathbb{R}$ *we have* $f(x+y) = f(x) + f(y)$

Problem 2 *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *which are continuous at* $x = 0$ *such that for all* $x, y \in \mathbb{R}$ *we have* $f(x+y) = f(x) + f(y)$

Problem 3 *Find all triples of continuous functions* f, g, h *such that for all* $x, y \in \mathbb{R}$ *we have* $f(x+y) = g(x) + h(y)$.

Problem 4 *Show that, if for a fixed* a *and any* x *we have*

$$f(x+a) = \frac{1+f(x)}{1-f(x)},$$

the function f *is periodic.*

Problem 5 *Find all polynomials* p *such that, for all* x, $p(x+1) = p(x) + 2x + 1$.

Problem 6 *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that for all* $x, y \in \mathbb{R}$ *we have*

$$xf(y) + yf(x) = (x+y)f(x)f(y)$$

Problem 7 *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) + f\left(\frac{1}{1-x}\right) = x \qquad \text{for all } x \neq 0, 1.$$

Problem 8 *Find all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *such that, for all* $x, y \in \mathbb{R}$, *we have*

$$f(x-y)f(x+y) = (f(x)f(y))^2$$

Problem 9 *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *which are continuous at* $x = 0$ *and such that for all* x, y *we have*

$$f(x+y) = f(x) + f(y) + xy(x+y)$$