Putnam training problems

2017 - Set 4

Problem 1 *Show that a among any $n+2$ integers either there are two whose sum is divisible by $2n$ or there are two whose difference is divisible by $2n$.*

Problem 2 *Show that for any set of $n+1$ integers, there is always a non-empty subset whose sum is divisible by n.*

Problem 3 *Show that among any nine points with integer coordinates in space there is always two of them such that the segment that joins them contains at least one more integer point.*

Problem 4 *All sides and diagonals of an octagon are colored either red or blue. Show that there are at least seven monochromatic triangles with vertices among the vertices of the octagon.*

Problem 5 *The each 1×1 cell of a 100×100 grid are colored with one of four possible colors in such a way that every row and every column has exactly 25 cells of each color. Show that there are four cells of the grid forming an axis-parallel rectangle which are colored with four different colors.*

Problem 6 *Show that if the plane is colored with 3 colors we can always find two points at distance 1 which have the same color.*

Problem 7 *Let $\triangle$ be an equilateral triangle in the plane. Show that if the points in the plane are colored red and blue, then there is either a segment of length one whose endpoints are both blue, or there is a translated copy of $\triangle$ whose vertices are all red.*

Problem 8 *Let $n = ab+1$, where a, b are positive integers. Show that in any ordered list of n different numbers there is either a sublist of length $a+1$ which is increasing or a sublist of length $b+1$ which is decreasing.*

Problem 9 *Let A be a set of n^2+1 lines in the plane, colored red, such that no two of them are parallel. Show that we can always find a vertical black line ℓ such that we can find a set of $n+1$ red lines whose intersections are all on the left side of ℓ, and a set of $n+1$ red lines whose intersections are all on the right side of ℓ.*

Problem 10 *Let $A_1, \ldots, A_{2n}$ be pairwise different subsets of a set with n elements. Find the minimum value of*

$$\sum_{i=1}^{2n} \frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}|},$$

where we consider $A_{2n+1} = A_1$.

Problem 11 *Let n and k be positive integers. Joe and José are going to write lists of integers. Joe writes down all the lists $a_1, \ldots, a_k$ such that $|a_1| + |a_2| + \ldots + |a_k| \leq n$. José writes down all the lists $b_1, \ldots, b_n$ such that $|b_1| + |b_2| + \ldots + |b_n| \leq k$. Prove that Joe and José wrote down the same number of lists.*