

Putnam training problems
2017 - Set 2

Problem 1. Find all polynomials $P(x)$ such that for all x

$$(x + 1)P(x) = (x - 10)P(x + 1)$$

Problem 2. Find all polynomials P for which there is a positive integer n such that for all x we have

$$P\left(x - \frac{1}{n}\right) + P\left(x + \frac{1}{n}\right) = 2P(x)$$

Problem 3. Let a, b, c, d, e be integers such that $a + b + c + d + e$ and $a^2 + b^2 + c^2 + d^2 + e^2$ are both divisible by some odd natural number n . Prove that $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$ is also divisible by n .

Problem 4. Find all polynomials P with integer coefficients such that for all x we have

$$P'(P(x)) = P(P'(x))$$

Problem 5. Find all polynomials P for which there is a natural number $n > 1$ such that for all real numbers x we have

$$P(x^n) = (P(x))^n$$

Problem 6. Let $P(x)$ be a polynomial with integer coefficients $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ and p be a prime number such that

- a_n is not divisible by p
- a_k is divisible by p for all $n - 1 \geq k \geq 0$.
- a_0 is not divisible by p^2 .

Prove that $P(x)$ cannot be written as the product of two polynomials with integer coefficients of degree at least 1 each.

Problem 7. Prove that $P(x) = x^{101} + 101x^{100} + 102$ cannot be written as the product of two polynomials with integer coefficients of degree at least 1 each.

Problem 8. Let $P(x)$ be a polynomial with integer coefficients. Let $k > 1$ be an integer. Consider

$$Q(x) = \underbrace{P(P(\dots(P(x))\dots))}_{k \text{ times}}$$

Show that if n is an integer and $Q(n) = n$ then $P(P(n)) = n$.

Problem 9. Let a_1, \dots, a_n be positive real numbers. Prove that the polynomial $P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \dots - a_{n-1} x - a_n$ has exactly one positive real root.