Putnam training problems

2017 - Set 2 hints

Problem 1. Find all polynomials $P(x)$ such that for all x

$$(x+1)P(x) = (x-10)P(x+1)$$

Hint: Show that $10, 9\dots, 0$ are roots of P. Then, show that P has no more roots.

Problem 2. Find all polynomials P for which there is a positive integer n such that for all x we have

$$P\left(x-\frac{1}{n}\right)+P\left(x+\frac{1}{n}\right)=2P(x)$$

Hint: If $a = P(0)$ and $b = P9\left(\frac{1}{n}\right)$, find a formula for $P\left(\frac{k}{n}\right)$ where $k \in \mathbb{N}$. Why must P be linear?

Problem 3. Let a, b, c, d, e be integers such that $a+b+c+d+e$ and $a^2+b^2+c^2+d^2+e^2$ are both divisible by some odd natural number n. Prove that $a^5+b^5+c^5+d^5+e^5-5abcde$ is also divisible by n. .

Hint: Let $P(x) = (x-a)(x-b)(x-c)(x-d)(x-e)$. Compute $P(a)+P(b)+P(c)+P(d)+P(e)$ in two different ways.

Problem 4. Find all polynomials P with integer coefficients such that for all x we have

$$P'(P(x)) = P(P'(x))$$

Hint: Look at the largest terms and compare.

Problem 5. Find all polynomials P for which there is a natural number $n > 1$ such that for all real numbers x we have

$$P(x^n) = (P(x))^n$$

Hint: Look at the second largest term.

Problem 6. Let $P(x)$ be a polynomial with integer coefficients $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ and p be a prime number such that

- a_n is not divisible by p
- a_k is divisible by p for all $n-1 \geq k \geq 0$.
- a_0 is not divisible by p^2.

Prove that $P(X)$ cannot be written as the product of two polynomials with integers coefficients of degree at least 1 each.

Hint: If the solution does not hold, then $P(x) = Q(x)R(x)$, with Q and R having integer coefficients. The constant terms of Q and R have a_0 as a product, which is multiple of p but not of p^2. That meanst that exactly one of those numbers is a multiple of p. If $R(x)$ is the one whose constant term is multiple of p, prove that all its terms are multiple of p. Why is this a contradiction?

Problem 7. Prove that $P(x) = x^{101} + 101x^{100} + 102$ cannot be written as the product of two polynomials with integer coefficients of degree at least 1 each.

Hint: Look at $P(x-1)$.

Problem 8. Let $P(x)$ be a polynomial with integer coefficients. Let $k > 1$ be an integer. Consider

$$Q(x) = \underbrace{P(P(\dots(P}_{k \text{ times}}(x))\dots)).$$

Show that if n is an integer and $Q(n) = n$ then $P(P(n)) = n$.

Hint: Prove that if a, b are integers, then $a - b$ divides $P(a) - P(b)$. If $Q(n) = n$, then consider the sequence $r_1 = P(n) - n$, $r_2 = P(P(n)) - P(n)$, $r_3 = P(P(P(n))) - P(P(n))$, ... What can you say about the absolute value of these numbers?

Problem 9. Let $a_1, \ldots, a_n$ be positive real numbers. Prove that the polynomial $P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \ldots - a_{n-1} x - a_n$ has exactly one positive real root.

Hint: If $P(x) = 0$ that means that $x = \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n}$. What do you know about the function on either side of that equation?