Putnam training problems

Set 1

Problem 1. Show that for any $n+1$ numbers in the set $\{1, 2, \ldots, 2n\}$ we can always find tow numbers such that one divides the other.

Problem 2. (Some Iberoamerican Mathematical Olympiad) Let $n > 10$ be a positive integer whose digits are only $1, 3, 7, 9$. Prove that n has a prime divisor which is larger than 10.

Problem 3. (Putnam 2016) Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

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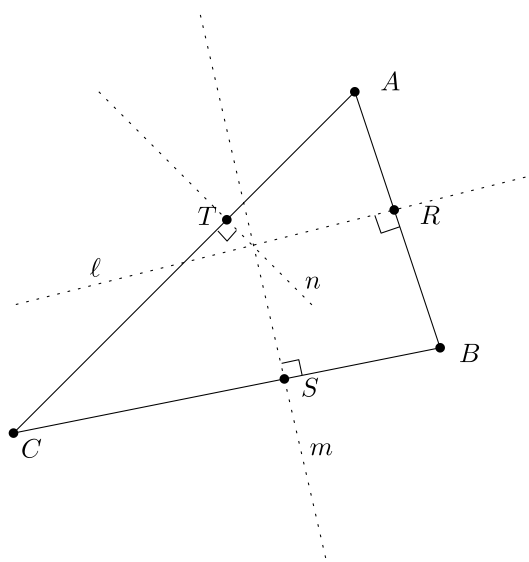
Problem 4. (Carnot's theorem) Let A, B, C be a the vertices of a triangle. Let R, S, T be points on the segments AB, BC, CA respectively. Let ℓ be the line through R orthogonal to AB, m the line through S orthogonal to BC and n the line through T orthogonal to CA. Show that ℓ, m, n concur (share a point) if and only if

$$AR^2 + BS^2 + CT^2 = TA^2 + RB^2 + SC^2.$$



Problem 5. (Putnam 2016) Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

Problem 6. (Putnam 2016) Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

Problem 7. You are the owner of an apartment complex. There are 120 apartments and 119 residents in total. We say that an apartment in *overpopulated* if at least 15 persons live in it. Every day, in some overpopulated apartment (if there is any), the residents have a fight a decided to all move out to othere apartments (which may have residents or not). Must this process always end?